

Ciągłe ułamki

Zadanie 1.

Rozwiń do ułamka ciągłego:

a) $3,14159265$

b) $\sqrt{101}$

c) $\sqrt{20}$

Rozwiązanie

$$\text{a) } 3,14159265 = 3 + 0,14159265 = 3 + \frac{14159265}{100000000} = 3 + \frac{2831853}{20000000} = 3 + \frac{1}{\frac{20000000}{2831853}} =$$

$$= 3 + \frac{1}{7 + \frac{177029}{2831853}} = 3 + \frac{1}{7 + \frac{1}{\frac{2831853}{177029}}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{176418}{177029}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{\frac{177029}{176418}}}} =$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{611}{176418}}}}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{\frac{176418}{611}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{450}{611}}}}}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{\frac{611}{450}}}}}} =$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{161}{450}}}}}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{\frac{450}{161}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{128}{2 + \frac{1}{161}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{161}{2 + \frac{1}{\frac{128}{161}}}}}}}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{33}{128}}}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{128}{33}}}}}}}}}} =$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{29}{33}}}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{33}{29}}}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{4}{29}}}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{29}{4}}}}}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{4}{29}}}}}}}}}}$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{29}}}}}}}}}} =$$

$$= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{288 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{7 + \frac{1}{4}}}}}}}}}}}}$$

b) $\sqrt{101} = \sqrt{100 + 1} = \sqrt{10^2 + 1}$

$$x = \sqrt{10^2 + 1}$$

$$x^2 = 10^2 + 1$$

$$x^2 - 10^2 = 1$$

$$(x - 10)(x + 10) = 1$$

$$x - 10 = \frac{1}{10 + x}$$

$$x = 10 + \frac{1}{10 + x}$$

$$x = 10 + \frac{1}{10 + \left(10 + \frac{1}{10 + x}\right)} = 10 + \frac{1}{10 + 10 + \frac{1}{10 + x}} = 10 + \frac{1}{20 + \frac{1}{10 + x}}$$

$$\begin{aligned}
 x &= 10 + \frac{1}{20 + \frac{1}{10+x}} = 10 + \frac{1}{20 + \frac{1}{10 + \left(10 + \frac{1}{20 + \frac{1}{10+x}}\right)}} = \\
 &= 10 + \frac{1}{20 + \frac{1}{10 + 10 + \frac{1}{20 + \frac{1}{10+x}}}} = 10 + \frac{1}{20 + \frac{1}{20 + \frac{1}{20 + \frac{1}{10+x}}}} \\
 x &= 10 + \frac{1}{20 + \frac{1}{20 + \frac{1}{20 + \frac{1}{20 + \frac{1}{\dots}}}}}
 \end{aligned}$$

c) $x = \sqrt{20}$

Ponieważ

$$4 < \sqrt{20} < 5$$

Więc

$$\sqrt{20} = 4 + \frac{1}{\alpha} \text{ gdzie } \alpha > 1$$

Zatem

$$\begin{aligned}
 \frac{1}{\alpha} &= \sqrt{20} - 4 \\
 \alpha &= \frac{1}{\sqrt{20} - 4} = \frac{\sqrt{20} + 4}{(\sqrt{20} - 4)(\sqrt{20} + 4)} = \frac{\sqrt{20} + 4}{20 - 16} = \frac{\sqrt{20} + 4}{4}
 \end{aligned}$$

Czyli

$$\begin{aligned}
 \sqrt{20} &= 4 + \frac{1}{\alpha} = 4 + \frac{1}{\frac{\sqrt{20} + 4}{4}} = 4 + \frac{1}{1 + \frac{\sqrt{20}}{4}} = 4 + \frac{1}{1 + \frac{4 + \frac{1}{\alpha}}{4}} = 4 + \frac{1}{1 + 1 + \frac{1}{\alpha}} = \\
 &= 4 + \frac{1}{2 + \frac{1}{4\alpha}} = 4 + \frac{1}{2 + \frac{1}{4 \cdot \left(\frac{\sqrt{20} + 4}{4}\right)}} = 4 + \frac{1}{2 + \frac{1}{\sqrt{20} + 4}} = \\
 &= 4 + \frac{1}{2 + \frac{1}{4 + 4 + \frac{1}{2 + \frac{1}{\sqrt{20} + 4}}}} = 4 + \frac{1}{2 + \frac{1}{8 + \frac{1}{2 + \frac{1}{\sqrt{20} + 4}}}}
 \end{aligned}$$

$$= 4 + \frac{1}{2 + \frac{1}{8 + \frac{1}{2 + \frac{1}{8 + \frac{1}{2 + \frac{1}{8 + \frac{1}{2 + \frac{1}{8 + \frac{1}{\dots}}}}}}}}}$$

Zadanie 2.

Znajdź granicę, do której dąży:

a) $3 + \frac{1}{6 + \frac{1}{6 + \frac{1}{6 + \frac{1}{\dots}}}}$

b) $5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\dots}}}}$

c) $1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{\dots}}}}}}}$

Rozwiązanie

a) Niech

$$x = 3 + \frac{1}{6 + \frac{1}{6 + \frac{1}{6 + \frac{1}{\dots}}}} = 3 + \frac{1}{3 + 3 + \frac{1}{6 + \frac{1}{6 + \frac{1}{\dots}}}} = 3 + \frac{1}{3 + x}$$

Zatem

$$\begin{aligned} x &= 3 + \frac{1}{3 + x} = \frac{9 + 3x + 1}{3 + x} = \frac{10 + 3x}{3 + x} \\ x^2 + 3x &= 10 + 3x \\ x^2 &= 10 \\ x &= \sqrt{10} \end{aligned}$$

b) Niech

$$x = 5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{5 + \frac{1}{\dots}}}}$$

Zatem

$$\begin{aligned} x &= 5 + \frac{1}{x} = \frac{5x + 1}{x} \\ x^2 &= 5x + 1 \\ x^2 - 5x - 1 &= 0 \\ \Delta &= 25 + 4 = 29 \end{aligned}$$

$$x = \frac{5 - \sqrt{29}}{2} \quad lub \quad x = \frac{5 + \sqrt{29}}{2}$$

Ponieważ chcemy, by $x > 0$, więc

$$x = \frac{5 + \sqrt{29}}{2}$$

c) Niech

$$x = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{\dots}}}}}}} = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{x}}} = 1 + \frac{1}{2 + \frac{1}{\frac{3x+1}{x}}} =$$

$$= 1 + \frac{1}{2 + \frac{x}{3x+1}} = 1 + \frac{1}{\frac{6x+2+x}{3x+1}} = 1 + \frac{1}{\frac{7x+2}{3x+1}} = 1 + \frac{3x+1}{7x+2} = \frac{7x+2+3x+1}{7x+2} = \frac{10x+3}{7x+2}$$

$$7x^2 + 2x = 10x + 3$$

$$7x^2 - 8x - 1 = 0$$

$$\Delta = 64 + 28 = 92 = 4 \cdot 23$$

$$\sqrt{\Delta}=2\sqrt{23}$$

$$x = \frac{8 - 2\sqrt{23}}{14} \quad lub \quad x = \frac{8 + 2\sqrt{23}}{14}$$

$$x = \frac{4 - \sqrt{23}}{7} \quad lub \quad x = \frac{4 + \sqrt{23}}{7}$$

Ponieważ $x > 0$, więc

$$x = \frac{4 + \sqrt{23}}{7}$$