Jacek Kredenc – szkic rozwiązania

Poszukiwania

Wyznacz wartości z dokładnością do pięciu miejsc po przecinku:

a) $\frac{1}{e}$; b) $\sqrt[3]{5}$; c) $\cos(1)$; d) $\sin\left(\frac{1}{5}\pi\right)$; e) $\tan(20°)$

Rozwiązanie

a)

$$f(x) = e^x$$

$$f^{(n)}(x) = \frac{1}{n!}e^x$$

Szukamy *n* dla którego: $f^{(n)}(0) \leq 0{,}00001$, czyli

$$\frac{1}{n!} \leq \frac{1}{100000}$$

Czyli

$$n! \geq 100000$$

$$100000: 2 = 50000$$

$$50000: 3 = 16666\frac{2}{3} < 16667$$

$$16667: 4 = 4166\frac{3}{4} < 4167$$

$$4167: 5 = 833\frac{2}{5} < 834$$

$$834: 6 = 139$$

$$139: 7 = 19\frac{6}{7} < 20$$

$$20: 8 = 2\frac{1}{2} < 3$$

$$3: 9 = \frac{1}{3}$$

Więc

$$n \geq 9$$

$$\frac{1}{e} = e^{-1} = e^0 + (e^0)' \cdot (-1) + \frac{1}{2}(e^0)'' \cdot (-1)^2 + \frac{1}{6}(e^0)''' \cdot (-1)^3 + \frac{1}{24}(e^0)^{IV} \cdot (-1)^4 +$$

$$+\frac{1}{120}(e^0)^V \cdot (-1)^5 + \frac{1}{720}(e^0)^{VI} \cdot (-1)^6 + \frac{1}{5040}(e^0)^{VII} \cdot (-1)^7 + \frac{1}{40320}(e^0)^{VIII} \cdot (-1)^8$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \frac{1}{40320} = 0{,}36788$$

Czyli $\frac{1}{e}$ z dokładnością do pięciu cyfr po przecinku wynosi 0,36788

b)

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

n=0

$$f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(8) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

Błąd

$$\left|\frac{1}{12} \cdot (5-8)\right| = \frac{1}{4} = 0{,}25$$

$\sqrt[3]{5} = 2$ z dokładnością do 0,25

n=1

$$\sqrt[3]{5} = 2 + \frac{1}{12} \cdot (5-8) = 2 + \frac{1}{12} \cdot (-3) = 2 - \frac{1}{4} = 1\frac{3}{4}$$

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$

$$f''(8) = \left(-\frac{2}{9}\right) \cdot \frac{1}{32} = -\frac{1}{144}$$

Błąd

$$\left|\left(-\frac{1}{144}\right) \cdot \frac{1}{2} \cdot 9\right| = \frac{1}{32} = 0{,}03125$$

$\sqrt[3]{5} = 1{,}75$ z dokładnością do 0,03125

n=2

$$\sqrt[3]{5} = 1{,}75 - 0{,}03125 = 1{,}71875$$

$$f'''(x) = \frac{10}{27} \cdot x^{-\frac{8}{3}}$$

$$f'''(8) = \frac{10}{27} \cdot \frac{1}{256} = \frac{5}{3456}$$

Błąd

$$\left|\frac{5}{3456}\cdot\frac{1}{6}\cdot(-27)\right| = \frac{5}{768} = 0{,}0065104166667$$

$\sqrt[3]{5} = 1{,}71875$ z dokładnością do 0,007

n=3

$$\sqrt[3]{5} = 1{,}71875 - \frac{5}{768} = 1{,}71223$$

$$f^{(IV)}(x) = -\frac{80}{81}\cdot x^{-\frac{11}{3}}$$

$$f^{(IV)}(8) = \left(-\frac{80}{81}\right)\cdot\frac{1}{2048} = -\frac{5}{10368}$$

Błąd

$$\left|\left(-\frac{5}{10368}\right)\cdot\frac{1}{24}\cdot 81\right| = \frac{5}{3072} = 0{,}0016276041667$$

$\sqrt[3]{5} = 1{,}71223$ z dokładnością do 0,002

n=4

$$\sqrt[3]{5} = 1{,}71223 - 0{,}0016276041667 = 1{,}71060$$

$$f^{(V)}(x) = \frac{880}{243}x^{-\frac{14}{3}}$$

$$f^{(V)}(8) = \frac{880}{243}\cdot\frac{1}{16384} = \frac{55}{248832}$$

Błąd

$$\left|\frac{11}{248832}\cdot\frac{1}{24}\cdot(-243)\right| = \frac{11}{24576} = 0{,}0004475911458$$

$\sqrt[3]{5} = 1{,}71060$ z dokładnością do 0,0005

n=5

$$\sqrt[3]{5} = 1{,}71060 - 0{,}0004475911458 = 1{,}71015$$

$$f^{(VI)}(x) = -\frac{12320}{729}x^{-\frac{17}{3}}$$

$$f^{(VI)}(8) = \left(-\frac{12320}{729}\right)\cdot\frac{1}{131072} = -\frac{385}{2985984}$$

Błąd

$$\left|\left(-\frac{385}{2985984}\right)\cdot\frac{1}{720}\cdot 729\right| = \frac{77}{589824} = 0{,}0001305474175$$

$\sqrt[3]{5} = 1{,}71015$ z dokładnością do 0,0002

n=6

$$\sqrt[3]{5} = 1{,}71015 - 0{,}0001305474175 = 1{,}71002$$

$$f^{(VII)}(x) = \frac{209440}{2187} x^{-\frac{20}{3}}$$

$$f^{(VII)}(8) = \frac{209440}{2187} \cdot \frac{1}{1048576} = \frac{6545}{71663616}$$

Błąd

$$\left| \frac{6545}{71663616} \cdot \frac{1}{5040} \cdot (-2187) \right| = \frac{1309}{33030144} = 0{,}0000396304660$$

$\sqrt[3]{5} = 1{,}71002$ z dokładnością do 0,00004

n=7

$$\sqrt[3]{5} = 1{,}71002 - 0{,}0000396304660 = 1{,}70998$$

$$f^{(VIII)}(x) = -\frac{4188800}{6561} x^{-\frac{23}{3}}$$

$$f^{(VIII)}(8) = \left(-\frac{4188800}{6561} \cdot \frac{1}{8388608} \right) = -\frac{32725}{429981696}$$

Błąd

$$\left| \left(-\frac{32725}{429981696} \cdot \frac{1}{40320} \cdot 6561 \right) \right| = \frac{6545}{528482304} = 0{,}0000123845206$$

$\sqrt[3]{5} = 1{,}70998$ z dokładnością do 0,00002

n=8

$$\sqrt[3]{5} = 1{,}70998 - 0{,}0000123845206 = 1{,}70997$$

$$f^{(IX)}(x) = \frac{96342400}{19683} x^{-\frac{26}{3}}$$

$$f^{(IX)}(8) = \frac{96342400}{19683} \cdot \frac{1}{67108864} = \frac{752675}{10319560704}$$

Błąd

$$\left| \frac{752675}{10319560704} \cdot \frac{1}{362880} \cdot (-19683) \right| = \frac{150535}{38050725888} = 0{,}0000039561663$$

$\sqrt[3]{5} = 1{,}70997$ z dokładnością do 0,00001 c)

$$f(x) = \cos x$$

Jeśli przyjmiemy, że $x_0 = 0$, to należy tak dobrać n, by

$$\frac{1}{n!} \leq 0{,}00001$$

Jak wyliczyliśmy w zadaniu a) warunek jest spełniony dla $n \geq 9$

Mamy więc

$$\cos 1 = \cos 0 + \cos' 0 + \frac{1}{2}\cos'' 0 + \frac{1}{6}\cos''' 0 + \frac{1}{24}\cos^{(IV)} 0 + \frac{1}{120}\cos^{(V)} 0 + \frac{1}{720}\cos^{(VI)} 0$$

$$+ \frac{1}{5040}\cos^{(VII)} 0 + \frac{1}{40320}\cos^{(VIII)} 0 = \cos 0 - \sin 0 - \frac{1}{2}\cos 0 + \frac{1}{6}\sin 0 + \frac{1}{24}\cos 0 -$$

$$-\frac{1}{120}\sin 0 - \frac{1}{720}\cos 0 + \frac{1}{5040}\sin 0 + \frac{1}{40320}\cos 0 = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40320} =$$

$$= 0{,}54030$$

d)

$$f(x) = \sin x$$

Jeśli przyjmiemy, że $x_0 = \frac{1}{6}\pi$, to należy tak dobrać n, by

$$\frac{1}{n!} \cdot \cos\left(\frac{\pi}{6}\right) \cdot \left(\frac{\pi}{30}\right)^n \leq 0{,}00001$$

$$\frac{1}{n!} \cdot \frac{\sqrt{3}}{2} \cdot \left(\frac{\pi}{30}\right)^n \leq \frac{1}{10000}$$

$$\frac{1}{n!} \cdot \left(\frac{\pi}{30}\right)^n \leq \frac{1}{10000}$$

$$\frac{1}{n!} \cdot \left(\frac{2}{15}\right)^n \leq \frac{1}{10000}$$

n=1

$$\frac{2}{15}$$

n=2

$$\frac{1}{2} \cdot \frac{4}{225} = \frac{2}{225}$$

n=3

$$\frac{1}{6} \cdot \frac{8}{3375} = \frac{4}{10125} = 0{,}00040$$

n=4

$$\frac{1}{24}\cdot\frac{16}{50625}=\frac{2}{151875}=0{,}000013$$

n=5

$\frac{1}{120}\cdot\frac{32}{759375}=\frac{4}{11390625}=0{,}0000004$

$$\sin\left(\frac{1}{5}\pi\right)=\sin\left(\frac{1}{6}\pi\right)+\sin'\left(\frac{1}{6}\pi\right)\cdot\frac{1}{30}\pi+\frac{1}{2}\cdot\sin''\left(\frac{1}{6}\pi\right)\cdot\frac{1}{900}\pi^2+$$

$$+\frac{1}{6}\sin'''\left(\frac{1}{6}\pi\right)\cdot\frac{1}{27000}\pi^3+\frac{1}{24}\sin^{(IV)}\left(\frac{1}{6}\pi\right)\cdot\frac{1}{810000}\pi^4=\sin\left(\frac{1}{6}\pi\right)+\cos\left(\frac{1}{6}\pi\right)\cdot\frac{1}{30}\pi-$$

$$-\frac{1}{2}\sin\left(\frac{1}{6}\pi\right)\cdot\frac{1}{900}\pi^2-\frac{1}{6}\cos\left(\frac{1}{6}\pi\right)\cdot\frac{1}{27000}\pi^3+\frac{1}{24}\sin\left(\frac{1}{6}\pi\right)\cdot\frac{1}{810000}\pi^4=\frac{1}{2}+$$

$$+\frac{\sqrt{3}}{2}\cdot\frac{1}{30}\pi-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{900}\pi^2-\frac{1}{6}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{27000}\pi^3+\frac{1}{24}\cdot\frac{1}{2}\cdot\frac{1}{810000}\pi^4=\frac{1}{2}+\frac{\sqrt{3}\pi}{60}-\frac{\pi^2}{3600}-$$

$$-\frac{\sqrt{3}\pi^3}{324000}+\frac{\pi^4}{38880000}=0{,}58779$$

e)

$$f(x)=tan(x)$$

$$x_0=30^\circ=\frac{1}{6}\pi$$

n=0

$$f\left(\frac{1}{6}\pi\right)=tan\left(\frac{1}{6}\pi\right)=\frac{\sqrt{3}}{3}=0{,}57735$$

Błąd

$$f'(x)=tan'(x)=\left(\frac{\sin x}{\cos x}\right)'=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=1+\tan^2 x$$

$$f'\left(\frac{1}{6}\pi\right)=1+\frac{1}{3}=1\frac{1}{3}$$

$$\frac{4}{3}\cdot\left(\frac{1}{9}\pi-\frac{1}{6}\pi\right)=\frac{4}{3}\cdot\left(-\frac{1}{18}\pi\right)=-\frac{2}{27}\pi\approx-\frac{8}{27}=-0{,}3$$

$\tan 20^\circ=0{,}57735$ z dokładnością do 0,3

n=1

$$\tan 20^\circ=\frac{\sqrt{3}}{3}-\frac{2}{27}\pi=0{,}34464$$

Błąd

$$f''(x) = (1 + \tan^2 x)' = 2\tan x(1 + \tan^2 x) = 2\tan x + 2\tan^3 x$$

$$f''\left(\frac{1}{6}\pi\right) = 2 \cdot \frac{\sqrt{3}}{3} + 2 \cdot \frac{\sqrt{3}}{9} = \frac{8\sqrt{3}}{9}$$

$$\frac{1}{2} \cdot \frac{8\sqrt{3}}{9} \cdot \left(-\frac{1}{18}\pi\right)^2 = \frac{4\sqrt{3}}{9} \cdot \frac{\pi^2}{324} = \frac{\sqrt{3}\pi^2}{729} \approx 0{,}05$$

$\tan 20° = 0{,}34464$z dokładnością do 0,05

n=2

$$\tan 20° = \frac{\sqrt{3}}{3} - \frac{2}{27}\pi + \frac{\sqrt{3}\pi^2}{729} = 0{,}36809$$

Błąd

$$f'''(x) = (2\tan x + 2\tan^3 x)' = 2(1 + \tan^2 x) + 6\tan^2 x(1 + \tan^2 x) =$$

$$= 2 + 2\tan^2 x + 6\tan^2 x + 6\tan^4 x = 2 + 8\tan^2 x + 6\tan^4 x$$

$$f'''\left(\frac{1}{6}\pi\right) = 2 + \frac{8}{3} + \frac{2}{3} = 5\frac{1}{3}$$

$$\frac{1}{6} \cdot \frac{16}{3} \cdot \left(-\frac{1}{18}\pi\right)^3 = \frac{8}{9} \cdot \left(-\frac{\pi^3}{5832}\right) \approx -\frac{64}{6561} = -0{,}01$$

$\tan 20° = 0{,}36809$ z dokładnością do 0,01

n=3

$$\tan 20° = \frac{\sqrt{3}}{3} - \frac{2}{27}\pi + \frac{\sqrt{3}\pi^2}{729} - \frac{\pi^3}{5832} = 0{,}36277$$

Błąd

$$f^{(IV)}(x) = (2 + 8\tan^2 x + 6\tan^4 x)' = 16\tan x(1 + \tan^2 x) + 24\tan^3 x(1 + \tan^2 x) =$$

$$= 16\tan x + 16\tan^3 x + 24\tan^3 x + 24\tan^5 x = 16\tan x + 40\tan^3 x + 24\tan^5 x =$$

$$= 8(2\tan x + 5\tan^3 x + 3\tan^5 x)$$

$$f^{(IV)}\left(\frac{1}{6}\pi\right) = 8\left(2 \cdot \frac{\sqrt{3}}{3} + 5 \cdot \frac{\sqrt{3}}{9} + 3 \cdot \frac{\sqrt{3}}{27}\right) = 8\left(\frac{6\sqrt{3}}{9} + \frac{5\sqrt{3}}{9} + \frac{\sqrt{3}}{9}\right) = \frac{32\sqrt{3}}{3}$$

$$\frac{1}{24} \cdot \frac{32\sqrt{3}}{3} \cdot \left(-\frac{1}{18}\pi\right)^4 = \frac{4\sqrt{3}}{9} \cdot \frac{\pi^4}{104976} = \frac{\sqrt{3}\pi^4}{236196} \approx \frac{512}{236196} = 0{,}003$$

$\tan 20° = 0{,}36277$ z dokładnością do 0,003

n=4

$$\tan 20° = \frac{\sqrt{3}}{3} - \frac{2}{27}\pi + \frac{\sqrt{3}\pi^2}{729} - \frac{\pi^3}{5832} + \frac{\sqrt{3}\pi^4}{236196} = 0{,}36349$$

Błąd

$$f^{(V)}(x) = (16\tan x + 40\tan^3 x + 24\tan^5 x)' =$$

$$= 16(1+\tan^2 x) + 120\tan^2 x(1+\tan^2 x) + 120\tan^4 x(1+\tan^2 x) =$$

$$= 16 + 16\tan^2 x + 120\tan^2 x + 120\tan^4 x + 120\tan^4 x + 120\tan^6 x =$$

$$= 16 + 136\tan^2 x + 240\tan^4 x + 120\tan^6 x$$

$$f^{(V)}\left(\frac{1}{6}\pi\right) = 16 + 45\frac{1}{3} + 26\frac{2}{3} + 4\frac{4}{9} = 92\frac{4}{9}$$

$$\frac{1}{120}\cdot\frac{832}{9}\cdot\left(-\frac{1}{18}\pi\right)^5 = \frac{104}{135}\cdot\left(-\frac{\pi^5}{1889568}\right) = -\frac{13\pi^5}{31886460} \approx -0{,}0005$$

$\tan 20° = 0{,}36349$ z dokładnością do 0,0005

n=5

$$\tan 20° = \frac{\sqrt{3}}{3} - \frac{2}{27}\pi + \frac{\sqrt{3}\pi^2}{729} - \frac{\pi^3}{5832} + \frac{\sqrt{3}\pi^4}{236196} - \frac{13\pi^5}{31886460} = 0{,}36336$$

Błąd

$$f^{(VI)}(x) = (16 + 136\tan^2 x + 240\tan^4 x + 120\tan^6 x)' =$$

$$= 272\tan x(1+\tan^2 x) + 960\tan^3 x(1+\tan^2 x) + 720\tan^5 x(1+\tan^2 x) =$$

$$= 272\tan x + 272\tan^3 x + 960\tan^3 x + 960\tan^5 x + 720\tan^5 x + 720\tan^7 x =$$

$$= 272\tan x + 1232\tan^3 x + 1680\tan^5 x + 720\tan^7 x$$

$$f^{(VI)}\left(\frac{1}{6}\pi\right) = \frac{272\sqrt{3}}{3} + \frac{1232\sqrt{3}}{9} + \frac{560\sqrt{3}}{9} + \frac{80\sqrt{3}}{9} = \frac{896\sqrt{3}}{3}$$

$$\frac{1}{720}\cdot\frac{896\sqrt{3}}{3}\cdot\left(-\frac{\pi}{18}\right)^6 = \frac{56\sqrt{3}}{135}\cdot\frac{\pi^6}{34012224} = \frac{7\sqrt{3}\pi^6}{573956280} \approx 0{,}0001$$

$\tan 20° = 0{,}36336$ z dokładnością do 0,0001

n=6

$$\tan 20° = \frac{\sqrt{3}}{3} - \frac{2}{27}\pi + \frac{\sqrt{3}\pi^2}{729} - \frac{\pi^3}{5832} + \frac{\sqrt{3}\pi^4}{236196} - \frac{13\pi^5}{31886460} + \frac{7\sqrt{3}\pi^6}{573956280} = 0{,}36338$$

Błąd

$$f^{(VII)}(x) = (272\tan x + 1232\tan^3 x + 1680\tan^5 x + 720\tan^7 x)' =$$

$$= 272(1+\tan^2 x) + 3696\tan^2 x(1+\tan^2 x) + 8400\tan^4 x\,(1+\tan^2 x) +$$

$$+5040\tan^6 x(1+\tan^2 x) =$$

$$= 272 + 272\tan^2 x + 3696\tan^2 x + 3696\tan^4 x + 8400\tan^4 x + 8400\tan^6 x +$$

$$+5040\tan^6 x + 5040\tan^8 x =$$

$$= 272 + 3968\tan^2 x + 12096\tan^4 x + 13440\tan^6 x + 5040\tan^8 x$$

$$f^{(VII)}\left(\frac{1}{6}\pi\right) = 272 + 1322\frac{2}{3} + 1344 + 497\frac{7}{9} + 62\frac{2}{9} = 3498\frac{2}{3}$$

$$\frac{1}{5040}\cdot\frac{10496}{3}\cdot\left(-\frac{\pi}{18}\right)^7 = -\frac{656}{945}\cdot\frac{\pi^7}{612220032} = -\frac{41\pi^7}{36159245640} \approx 0{,}00002$$

$\tan 20° = 0{,}36338$ z dokładnością do 0,00002

n=7

$$\tan 20° = \frac{\sqrt{3}}{3} - \frac{2}{27}\pi + \frac{\sqrt{3}\pi^2}{729} - \frac{\pi^3}{5832} + \frac{\sqrt{3}\pi^4}{236196} - \frac{13\pi^5}{31886460} + \frac{7\sqrt{3}\pi^6}{573956280} - \frac{41\pi^7}{36159245640} = 0{,}36338$$